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# Metric study, first Chern class on $G_{2,4}\mathbb{C}$ and special lagrangian manifolds on $G_{p,p+q}\mathbb{C}$

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Abstract. We know that the potential of the metric on $\mathbb{P}_m\mathbb{C}$ is equal to $(m+1)$. In this paper we give the metric potential of $G_{2,4}\mathbb{C}$ and an example of special lagrangian submanifold on a real locus of a Calabi-Yau manifold in $G_{p,p+q}\mathbb{C}$ which vanishing first Chern class.

**Keywords:** Grassmann manifold, First Chern class, Special Lagrangian manifolds.

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## Introduction

In geometry, Plücker coordinates, introduced by Julius Plücker in the 19th century, are a way to assign six homogenous coordinates to each line in projective 3-space, $\mathbb{P}_3$. Because they satisfy a quadratic constraint, they establish a one-to-one correspondence between the 4-dimensional space of lines in $\mathbb{P}_3$ and points on a quadric in $\mathbb{P}_5$ (projective 5-space). A predecessor and special case of Grassmann coordinates (which describe $k$-dimensional linear subspaces, or flats, in an $n$-dimensional Euclidean space), Plücker coordinates arises naturally in geometric algebra. They have proved useful for computer graphics, and also can be extended to coordinates for the screws and wrenches in the theory of kinematics used for robot control. Alternatively, let $L$ be a line contained in distinct planes a and b with homogeneous coefficients ($(a_0 : a_1 : a_2 : a_3)$ and $(b_0 : b_1 : b_2 : b_3)$. Let $N$ be the $2 * 4$ matrix with these coordinates as rows.

$$N = \begin{pmatrix} a_0 & a_1 & a_2 & a_3 \\ b_0 & b_1 & b_2 & b_3 \end{pmatrix}$$

We define dual Plücker coordinate $p_{ij}$ as the determinant of columns $i$ and $j$ of $N$,

$$p^{ij} = a_i b_j - a_j b_i.$$

Dual coordinates are convenient in some computations, and we can show that they are equivalent to primary coordinates. Specifically, let $(i, j, k, l)$ be an even permutation of $(0, 1, 2, 3)$; then

$$p_{ij} = p^{kl}.$$

- the symplectic form $\omega$ associated to the kählerian structure restricted to the $n$ real dimensional submanifold $L$ must identically vanish, i.e. $i^*\omega = 0$, where $i : L \to M$ is the canonical injection. So the maximal isotropic submanifold $L$, is endowed by a Lagrangian structure.
- There exists a $(n, 0)$-holomorphic volume form $\Omega$ such that

  $$\Omega \wedge \overline{\Omega} = \frac{2^n(-i)^{n^2}}{n!}\omega^n.$$

  This last one is given by the solution of Calabi conjecture.
- Finally, $L$ is a special Lagrangian manifold if we have $i^*\Omega = dV_L$ (in the general case we have $i^*\Omega = \lambda dV_L$ where $\lambda \in S^1$).

This paper gives an example of a complex hypersurface of the Grassmannians $G_{2,5}\mathbb{C}$ respectively $G_{p,p+q}\mathbb{C}$ with vanishing $C_1$. It is well known that its real locus is a special Lagrangian orbifold. Actually, it is a complete intersection of a transveral curves quartic in $\mathbb{P}_9\mathbb{C}$ respectively $\mathbb{P}_{C^p_{p+q}-1}$ . The first interpretation (hypersurface of $G_{2,5}\mathbb{C}$) does not require particular skills in Algebraic Geometry. Wherefore, both approaches will be developed in this article. The first section of this paper is devoted to the description of the manifold $X$ (definition and calculus of its first Chern class). In the second part, we stand and prove the main result in any dimension, giving both interpretations of the real locus $L$ of $X$.

## 1. Metric study and first Chern class in complex grassmannian $G_{2,4}\mathbb{C}$

### 1.1. **Definition of $G_{2,4}\mathbb{C}$.**

$(G_{2,4}\mathbb{C}, g)$ is the complex Grassmannian of two planes $\mathbb{C}^4$ equipped with the metric $g$, obtained from the Fubini-Study on $\mathbb{P}_5\mathbb{C}$, belonging to the first Chern class, $c_1(G_{2,4}\mathbb{C})$.
Points are identified $G_{2,4}\mathbb{C}$, a matrix $M_{4,2}(\mathbb{C})$:

$$\begin{pmatrix} z_0 & z'_0 \\ z_1 & z'_1 \\ z_2 & z'_2 \\ z_3 & z'_3 \end{pmatrix}$$

where the two column vectors are independent. The open cards $U_{i,j}$, $0 \leq i < j \leq 3$ are obtained by considering the minor of order on lines $i, j$ with nonzero determinant. For example, $U_{0,1}$, a point of the Grassmannian is:

$$\begin{pmatrix} 1 & 0 \\ 0 & 1 \\ z_1 & z_2 \\ z_3 & z_4 \end{pmatrix},$$

## 2. The first Chern class of $G_{2,4}\mathbb{C}$

The metric is written as follows:

$$g_{\lambda\bar{\mu}} = a_{2,4}\partial_{\lambda\bar{\mu}} Ln(K)$$

with $K$ is the potential of the metric, in effecting a change of map we obtain potential

$$\tilde{K} = |Jac|^2 K$$

Where $Jac$ is the Jacobian of change cards. In fact:

There are two types of change cards in $G_{2,4}\mathbb{C}$. The first and the easiest type is:

$$(\zeta_1, \zeta_2, \zeta_3, \zeta_4) \longrightarrow (-\frac{\zeta_2}{\zeta_1}, \frac{1}{\zeta_1}, -\frac{D}{\zeta_1}, \frac{\zeta_3}{\zeta_1}),$$

there $D = (\zeta_1\zeta_4 - \zeta_2\zeta_3)$.The determinant of such a change of cards is equal to $1/\zeta_1^4$ which can extend the expression to the new card. The second change is the type of card :

$$(\zeta_1, \zeta_2, \zeta_3, \zeta_4) \longrightarrow (-\frac{\zeta_4}{D}, \frac{\zeta_2}{D}, \frac{\zeta_3}{D}, -\frac{\zeta_1}{D}).$$

Its determinant (much more complicated to calculate) is equal to $1/D^4$, which still allows extended writing the new card. In fact:

$$(z_0, z_2, z_3, z_4) \rightarrow \begin{pmatrix} 1 & 0 \\ z_1 & z_2 \\ 0 & 1 \\ z_3 & z_4 \end{pmatrix}$$

$$\begin{pmatrix} 1 & 0 \\ z_1 & z_2 \\ 0 & 1 \\ z_3 & z_4 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ z_1 & z_2 \end{pmatrix}^{-1} = \frac{1}{z_2} \begin{pmatrix} 1 & 0 \\ z_1 & z_2 \\ 0 & 1 \\ z_3 & z_4 \end{pmatrix} \begin{pmatrix} z_2 & 0 \\ -z_1 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ -\frac{z_1}{z_2} & \frac{1}{z_2} \\ \frac{z_2 z_3 - z_1 z_4}{z_2} & \frac{z_4}{z_2} \end{pmatrix}$$

$$(z_1, z_2, z_3, z_4) \rightarrow (-\frac{z_1}{z_2}, \frac{1}{z_2}, \frac{z_2 z_3 - z_1 z_4}{z_2}, \frac{z_4}{z_2})$$

$$\begin{vmatrix} -\frac{1}{z_2} & \frac{z_1}{z_1^2} & 0 & 0 \\ 0 & \frac{1}{z_2^2} & 0 & 0 \\ -\frac{z_4}{z_2} & \frac{z_1 z_4}{z_2^2} & 1 & -\frac{z_1}{z_2} \\ 0 & -\frac{z_4}{z_2^2} & 0 & \frac{1}{z_2} \end{vmatrix} = \frac{1}{|z_2|^4} \Rightarrow |Jac|^2 = \frac{1}{|z_2|^{2\times 4}}.$$

Or potential following metric:

$$K=(1+|z_1|^2+|z_2|^2+|z_3|^2+|z_4|^2+|z_1z_4-z_2z_3|^2)^a$$

by making the change of maps defined above, we get:

$$\tilde{K}=(1+\left|\frac{z_1}{z_2}\right|^2+\left|\frac{1}{z_2}\right|^2+\left|\frac{z_1z_4-z_2z_3}{z_2}\right|^2+\left|\frac{z_4}{z_2}\right|^2+\left|\frac{z_1z_4}{z_2^2}+\frac{z_2z_3-z_1z_4}{z_2^2}\right|)^a$$

which gives:

$$\tilde{K}=\frac{1}{|z_2|^{2a}}(1+|z_1|^2+|z_2|^2+|z_3|^2+|z_2|^2+|z_1z_4-z_2z_3|^2)^a\Rightarrow a=4$$

where $c_1(G_{2,4}\mathbb{C})>0$ for coefficient $a_{2,4}=4$ is defined by the following metrics:

$$g_{\lambda\overline{\mu}}=a_{2,4}\partial\overline{\partial}\ln(1+|z_1|^2+|z_2|^2+|z_3|^2+|z_4|^2+|z_1z_4-z_2z_3|^2).$$

## 3. Definition and description of the manifold $X$

Let us consider the following equation:

$$\begin{aligned} F(u,v) &= (z_0z_1'-z_1z_0')^4+(z_0z_2'-z_2z_0')^4+(z_0z_3'-z_3z_0')^4 \\ &- (z_1z_2'-z_2z_1')^4-(z_1z_3'-z_3z_1')^4-(z_2z_3'-z_3z_2')^4=0, \end{aligned} \tag{3.1}$$

where $u=(z_0,z_1,z_2,z_3)$ and $u'=(z_0',z_1',z_2',z_3')$ are two independent vectors of $\mathbb{C}^4$. It is easy to see that this equation depends only on the 2-plane of $\mathbb{C}^4$ given by $u$ and $u'$. Consequently, (1) defines a subset of the grassmannian $G_{2,4}\mathbb{C}$, set of complex two dimensional linear spaces of $\mathbb{C}^4$.

Lemma 3.1. *The equation (1) defines a holomorphic hypersurface of $G_{2,4}\mathbb{C}$ which may be identified with a complete intersection of a quadric and a quartic in $\mathbb{P}_5\mathbb{C}$.*

*Proof.* In order to find the rank of the linear tangent map of $F$ in a $G_{2,4}\mathbb{C}$ classical coordinates system $(\zeta_i)_{i\in\{1,..,4\}}$, we consider :

$$\left\{\begin{array}{r} \zeta_1^3-\zeta_4\left(\zeta_1\zeta_4-\zeta_2\zeta_3\right)^3=0 \\ \zeta_2^3+\zeta_3\left(\zeta_1\zeta_4-\zeta_2\zeta_3\right)^3=0 \\ -\zeta_3^3+\zeta_2\left(\zeta_1\zeta_4-\zeta_2\zeta_3\right)^3=0 \\ -\zeta_4^3-\zeta_1\left(\zeta_1\zeta_4-\zeta_2\zeta_3\right)^3=0 \end{array}\right. \tag{3.2}$$

Using equation (1), we firstly prove that the solutions of this system are necessarily of norm one. Taking into account all different cases, explicit computations lead to a contradiction with the system (2).

Let us give another description of $X$. To this end, we use the classical identification of $G_{2,4}\mathbb{C}$ with the quadric in $\mathbb{P}_5\mathbb{C}$ given by :

$$\eta_0\eta_5-\eta_1\eta_4+\eta_2\eta_3=0,$$

via the map which associates to every point

$$\begin{pmatrix} z_0 & z_0' \\ z_1 & z_1' \\ z_2 & z_2' \\ z_3 & z_3' \end{pmatrix}\in G_{2,4}\mathbb{C}.$$

described in the above coordinates, the point of $\mathbb{P}_5\mathbb{C}$ :

$$[\eta_0 = z_0z_1' - z_1z_0', \eta_1 = z_0z_2' - z_2z_0', \eta_2 = z_0z_3' - z_3z_0',$$
$$\eta_3 = z_1z_2' - z_2z_1', \eta_4 = z_1z_3' - z_3z_1', \eta_5 = z_2z_3' - z_3z_2'].$$

So, $X$ appears as a complete intersection of a quadric and a quartic in $\mathbb{P}_5\mathbb{C}$, given by :

$$\left\{ \begin{array}{c} \eta_0\eta_5 - \eta_1\eta_4 + \eta_2\eta_3 = 0 \\ \eta_0^4 + \eta_1^4 + \eta_2^4 - \eta_3^4 - \eta_4^4 - \eta_5^4 = 0). \end{array} \right. \tag{3.3}$$

LEMMA 3.2. *$X$ is a manifold with vanishing first Chern class.*

*Proof.* In a first time, we shall give a "self-contained" proof, using the description of $X$, given by equation (1). Then, in a second time, taking into account equation (3), we give a shorter proof which uses some concepts of Algebraic Geometry.

1) Let us prove that the determinant bundle $\Lambda^3T^*X$ is trivializable by giving a $(3,0)$-holomorphic volume form $\Omega$ from a $(4,0)$-meromorphic form on $G_{2,4}\mathbb{C}$, using Poincaré residue. It corresponds to the generalization of the classical Cauchy residue at a point of a domain of $\mathbb{C}$ to the concept of residue in a hypersurface of a $n$ dimensional complex manifold. If $\eta$ is a $n$-meromorphic form of such a manifold which has first order poles on the hypersurface $X$ locally defined by the equation $g = 0$, then $\eta$ may be locally written as:

$$\eta = \frac{\gamma \wedge dg}{g} + \delta,$$

where $\gamma$ and $\delta$ are respectively $(n-1)$ and $n$ holomorphic forms. Then, the restriction of $\gamma$ to $X$ is well defined as a $(n-1)$ holomorphic form on $X$. We say that $\gamma$ is the Poincaré residue of $\eta$.

In our case, let us consider the open chart set $U_{01}$ of $G_{2,4}\mathbb{C}$ where the $\{U_{ij}\}_{0\le i<j\le 3}$ are given by :

$$U_{ij} = \{(u,v) \in \mathbb{C}^4 \times \mathbb{C}^4 : u = (z_0,z_1,z_2,z_3), u' = (z_0',z_1',z_2',z_3'), z_iz_j' - z_jz_i' \neq 0\}.$$

The chart maps $\varphi_{ij} : U_{ij} \to \mathbb{C}^4 \sim M_2(\mathbb{C})$, are defined, similarly to $\varphi_{01}$ in the following manner :

$$\varphi_{01}(u,v) = \begin{pmatrix} z_2 & z_2' \\ z_3 & z_3' \end{pmatrix} \times \begin{pmatrix} z_0 & z_0' \\ z_1 & z_1' \end{pmatrix}^{-1} = \begin{pmatrix} \zeta_3 & \zeta_1 \\ \zeta_4 & \zeta_2 \end{pmatrix}$$

The expression of $F$ in $U_{01}$ is given by

$$f_{01}(\zeta_1,\zeta_2,\zeta_3,\zeta_4) = 1 + \zeta_1^4 + \zeta_2^4 - \zeta_3^4 - \zeta_4^4 - (\zeta_1\zeta_4 - \zeta_2\zeta_3)^4,$$

In $U_{01}$, let us consider :

$$\eta = \frac{d\zeta_1 \wedge d\zeta_2 \wedge d\zeta_3 \wedge d\zeta_4}{1 + \zeta_1^4 + \zeta_2^4 - \zeta_3^4 - \zeta_4^4 - (\zeta_1\zeta_4 - \zeta_2\zeta_3)^4}.$$

$\eta$ is the local expression in the open chart set $U_{01}$ of a $n$ meromorphic form globally defined on $G_{2,4}\mathbb{C}$ whose poles are along $X$. Indeed, the power 4 is the correct one.

This can easily be seen in proceeding to the change of charts. There are two types of change of charts in $G_{2,4}$. The first ones (or the easiest) are of the form:

$$(\zeta_1, \zeta_2, \zeta_3, \zeta_4) \longrightarrow (-\frac{\zeta_2}{\zeta_1}, \frac{1}{\zeta_1}, -\frac{D}{\zeta_1}, \frac{\zeta_3}{\zeta_1}),$$

where $D = (\zeta_1\zeta_4 - \zeta_2\zeta_3)$. The determinant of such a change of charts is equal to $1/\zeta_1^4$, and so $\eta$ may be extended to the new chart. The second change of charts is:

$$(\zeta_1, \zeta_2, \zeta_3, \zeta_4) \longrightarrow (-\frac{\zeta_4}{D}, \frac{\zeta_2}{D}, \frac{\zeta_3}{D}, -\frac{\zeta_1}{D}).$$

Its determinant (much more difficult to compute) is equal to $1/D^4$, so, we may extend the expression to the new chart. This result has been established in the general case of $G_{p,p+q}\mathbb{C}$. The Poincaré residue of

$$\eta = \frac{d\zeta_1 \wedge d\zeta_2 \wedge d\zeta_3 \wedge d\zeta_4}{1 + \zeta_1^4 + \zeta_2^4 - \zeta_3^4 - \zeta_4^4 - (\zeta_1\zeta_4 - \zeta_2\zeta_3)^4}$$

is given, in a submersion chart $X$ obtained as a sub-chart of $(U_{01}, \varphi_{01})$ considering the condition $\partial f_{01}/\partial\zeta_1 \neq 0$, by the expression:

$$\gamma = \frac{-d\zeta_2 \wedge d\zeta_3 \wedge d\zeta_4}{\partial f_{01}/\partial\zeta_1},$$

which defines a holomorphic 3 volume form on $X$. This proves that $C_1(X) = 0$.
2) Using the description (3) of $X$, we can directly establish this last result. Actually, a similar proof to the preceding shows that a hypersurface of degree $m+1$ of $\mathbb{P}_m\mathbb{C}$ has necessarily a vanishing $C_1$. Taking into account that the sum of the degrees of a quadric and a quartic is equal to 6, and because our intersection is complete in $\mathbb{P}_5\mathbb{C}$, we obtain the result thanks to the adjunction formula.

## 4. Special lagrangian submanifold

Theorem 4.1. *The real locus $L$ of $X$ is a special Lagrangian submanifold endowed with a circle bundle structure over $\mathbb{P}_2\mathbb{R}$ which may be identified with $S^3/\mathbb{Z}_4$.*

*Proof.* In order to show that $L$ is a special Lagrangian submanifold, we use the result given by R.L. Bryant [2]: the real locus of a trivial first Chern class manifold (that is to say the fixed points of an anti-holomorphic involution), when it is non-empty, is a special Lagrangian manifold which is, according to the lemmas 3.1 and 3.2, the case of the manifold $X$ (the anti-holomorphic involution we use here is the classical conjugation). $L = L'/\sim$, where $L'$ is the set of the 2-planes of $\mathbb{R}^4$ seen as pairs of independent vectors $(u, u') \in \mathbb{R}^4 \times \mathbb{R}^4$, $u = (x_0, x_1, x_2, x_3), u' = (x'_0, x'_1, x'_2, x'_3)$ (defining a point of $G_{2,4}\mathbb{R}$) such that:

$$\begin{cases} (x_0x'_1 - x_1x'_0)^4 + (x_0x'_2 - x_2x'_0)^4 + (x_0x'_3 - x_3x'_0)^4 = 1 & (E_1) \\ (x_1x'_2 - x_2x'_1)^4 + (x_1x'_3 - x_3x'_1)^4 + (x_2x'_3 - x_3x'_2)^4 = 1 & (E_2) \end{cases}$$

and $\sim$ is the equivalence relation defined by: $(u, u') \sim (v, v') \in \mathbb{R}^4 \times \mathbb{R}^4$, if and only if $v = au + cu'$ and $v' = bu + du'$ with $ad - bc = \pm 1$.

$(E_1)$ and $(E_2)$ come from the equation (1) and a given normalization. So $L'$ is a submanifold of $\mathbb{R}^8$, diffeomorphic to the pull-back of $(1,1) \in \mathbb{R}^2$ by the submersion

$$\psi : (\mathbb{R} \times \mathbb{R}^3) \times (\mathbb{R} \times \mathbb{R}^3) \to \mathbb{R}^2$$

defined by

$$\psi((\alpha, u_0), (\alpha', {u_0}')) = (\|u_0 \wedge {u_0}'\|^2, \|\alpha' u_0 - \alpha {u_0}'\|^2).$$

In the former description $u = (\alpha, u_0) \in \mathbb{R}^4$ where $\alpha = x_0$ and $u_0 = (x_1, x_2, x_3) \in \mathbb{R}^3$ is the projection of $u = (x_0, x_1, x_2, x_3)$ on $\mathbb{R}^3 = \{(0, x, y, z) \in \mathbb{R}^4\}$ (it is the same for the prime items). So the expression $u_0 \wedge {u_0}'$ is intrinsic. The result is more difficult to obtain for $\|\alpha' u_0 - \alpha {u_0}'\|^2$, after quotienting by $\sim$. Indeed, if

$$v = au + cu' = (\beta, v_0) \text{ and } v' = bu + du' = (\beta', {v_0}') \text{ with } ad - bc = \pm 1,$$

we have

$$\beta = a\alpha + c\alpha', \quad \beta' = b\alpha + d\alpha', \;\; v_0 = au_0 + c{u_0}' \text{ and } {v_0}' = bu_0 + d{u_0}'.$$

So

$$\begin{aligned} \beta' v_0 - \beta {v_0}' &= (b\alpha + d\alpha')(au_0 + c{u_0}') - (a\alpha + c\alpha')(bu_0 + d{u_0}') \\ &= (ad - bc)(\alpha' u_0 - \alpha {u_0}') \\ &= \pm(\alpha' u_0 - \alpha {u_0}'). \end{aligned}$$

The interpretation of $L$ as a circle bundle over $\mathbb{P}_2\mathbb{R}$ may be realized as follows:

- $(E_2)$ determines the basis of this bundle and corresponds to the fact that the vectorial product of the vectors projections $u = (x_0, x_1, x_2, x_3)$ and $u' = (x_0', x_1', x_2', x_3')$ on $\mathbb{R}^3 = \{(0, x, y, z) \in \mathbb{R}^4\}$ is unitary for a certain norm. According to $(E_2)$ the vectors $u_0 = (0, x_1, x_2, x_3)$ and ${u_0}' = (0, x_1', x_2', x_3')$ are linearly independent. Then they define a point of the Grassmannian $G_{2,3}\mathbb{R}$, that is to say a point of $\mathbb{P}_2\mathbb{R}$ (given by their vectorial product).
- $(E_1)$ describes the fibres above $\mathbb{P}_2\mathbb{R}$. To any pair $(w, w')$ of independent vectors of $\mathbb{R}^3$, corresponds the circle in the basis $\{w, w'\}$ given by the equation $\|\alpha' w - \alpha w'\|^2 = 1$.

Using the second interpretation of $X$ (intersection of a quadric and a quartic), and the notations used in the equations (3.3), $L$ may be identified with $S^3/\mathbb{Z}_4$ in the following way : It is the set of the vectors pairs of $\mathbb{R}^3$ $(u, v)$ where $u = (\eta_0, \eta_1, \eta_2)$ and $v = (\eta_5, -\eta_4, \eta_3)$ with vanishing scalar product fulfilling the normalization condition

$$\eta_0^4 + \eta_1^4 + \eta_2^4 = \eta_5^4 + (-\eta_4)^4 + \eta_3^4 = 1,$$

which is topologically equivalent to $\|u\| = \|v\|$. If one considers the unit vectors $u' = u/\|u\|$ and $v' = v/\|v\|$, the triple $(u', v', u' \wedge v')$ is a direct orthonormal basis of $\mathbb{R}^3$ which can be viewed as a matrix in $SO(3)$. The one to one correspondence

$$[\eta_0, .., \eta_5] \longrightarrow \{(u', v'), (-u', -v')\},$$

defines a map from $L$ in $SO(3)/\mathbb{Z}_2$, where $\mathbb{Z}_2$ is the subgroup $\{Id, \sigma\}$ of $SO(3)$ generated by the matrix $\sigma = \begin{pmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$, which corresponds to the map

$$(u', v', u' \wedge v') \longrightarrow (-u', -v', u' \wedge v').$$

This map being a diffeomorphism, $L$ is diffeormorphic to $SO(3)/\mathbb{Z}_2$ or equivalently $SU(2)/\mathbb{Z}_4$ that is $S^3/\mathbb{Z}_4$, using the double cover $SU(2) \longrightarrow SO(3)$. We can recover the description of $L$ (given in 1)) as a circle bundle over $\mathbb{P}_2\mathbb{R}$ by projecting

$$\begin{array}{ccc} SO(3)/\mathbb{Z}_2 & \longrightarrow & \mathbb{P}_2\mathbb{R} \\ \{(u', v', u' \wedge v'), (-u', -v', u' \wedge v')\} & \longrightarrow & \{u', -u'\}. \end{array}$$

## 5. Definition and description of the hypersurface $Y$

Let us consider the following equation:

$$F(u, v) \quad = \quad \sum_{i=0}^{3} \sum_{j=i+1}^{4} (z_i z'_j - z'_i z_j)^n = 0 \tag{5.1}$$

where $u = (z_0, z_1, z_2, z_3, z_4)$ and $u' = (z'_0, z'_1, z'_2, z'_3, z'_4)$ are two independent vectors of $\mathbb{C}^5$. It is easy to see that this equation depends only on the 2-plane of $\mathbb{C}^5$ given by $u$ and $u'$. Consequently, (5.1) defines a subset of the grassmannian $G_{2,5}\mathbb{C}$, set of complex two dimensional linear spaces of $\mathbb{C}^5$. Let give the principle results of this paper:

### Lemma

*The equation (1) defines a holomorphic hypersurface of $G_{2,5}\mathbb{C}$ which may be identified with a complete intersection of a cross transversal curves in $\mathbb{P}_9\mathbb{C}$.*

### Proof

We showed that this equation does not depend on the choice of the representatives $u$ and $u'$ of 2 - plane of $\mathbb{C}^5$. Indeed, if $v$ and $v'$ are two other representatives) the identities: $v = au + bu'$ and $v' = cu + du'$.

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in GL(2, \mathbb{C}).$$

$$F(u', v') \quad = \quad (ad - bc)^n F(u, v) \tag{5.2}$$

$U_{ij} = \{(u, v) \in \mathbb{C}^5 \times \mathbb{C}^5 : u = (z_0, z_1, z_2, z_3, z_4), u' = (z'_0, z'_1, z'_2, z'_3, z'_4), z_i z'_j - z_j z'_i \neq 0\}$.
The applications of maps $\varphi_{ij} : U_{ij} \to \mathbb{C}^5$, are defined following the example of $\varphi_(01 - plane)$ in the following way:

$$\varphi_{01}(u, v) = \begin{pmatrix} z_2 & z'_2 \\ z_3 & z'_3 \\ z_4 & z'_4 \end{pmatrix} \times \begin{pmatrix} z_0 & z'_0 \\ z_1 & z'_1 \end{pmatrix}^{-1} = \begin{pmatrix} \xi_1 & \xi_2 \\ \xi_3 & \xi_4 \\ \xi_5 & \xi_6 \end{pmatrix}$$

The application $f_{01} : U_{01} \subset G_{2,5}\mathbb{C} \to \mathbb{C}$, which, in a plan engendered by $u = (z_0, z_1, z_2, z_3, z_4)$ and $u' = (z'_0, z'_1, z'_2, z'_3, z'_4)$ associate $1 + \sum_{i=1} \xi_i^n + (\xi_1\xi_4 - \xi_2\xi_3)^n + (\xi_1\xi_6 - \xi_2\xi_5)^n + (\xi_3\xi_6 - \xi_4\xi_5)^n$, where

$$\varphi_{01}(u, v) = \begin{pmatrix} \xi_1 & \xi_2 \\ \xi_3 & \xi_4 \\ \xi_5 & \xi_6. \end{pmatrix}$$

In order to find the rank of the linear tangent map of $F$ gave by

$$df_{01}(\xi_1, \xi_2, \xi_3, \xi_4, \xi_5, \xi_6) \neq 0$$

in a $G_{2,5}\mathbb{C}$ classical coordinates system $(\xi_i)_{i\in\{1,..,6\}}$, we consider :

$$\left\{ \begin{array}{l} \xi_1^{n-1} + \xi_4(\xi_1\xi_4 - \xi_2\xi_3)^{n-1} + \xi_6(\xi_1\xi_6 - \xi_2\xi_5)^{n-1} = 0 \\ \xi_2^{n-1} - \xi_3(\xi_1\xi_4 - \xi_2\xi_3)^{n-1} - \xi_5(\xi_1\xi_6 - \xi_2\xi_5)^{n-1} = 0 \\ \xi_3^{n-1} - \xi_2(\xi_1\xi_4 - \xi_2\xi_3)^{n-1} + \xi_6(\xi_3\xi_6 - \xi_4\xi_5)^{n-1} = 0 \\ \xi_4^{n-1} + \xi_1(\xi_1\xi_4 - \xi_2\xi_3)^{n-1} - \xi_5(\xi_3\xi_6 - \xi_4\xi_5)^{n-1} = 0 \\ \xi_5^{n-1} - \xi_2(\xi_1\xi_6 - \xi_2\xi_5)^{n-1} - \xi_4(\xi_3\xi_6 - \xi_4\xi_5)^{n-1} = 0 \\ \xi_6^{n-1} + \xi_1(\xi_1\xi_6 - \xi_2\xi_5)^{n-1} + \xi_3(\xi_3\xi_6 - \xi_4\xi_5)^{n-1} = 0. \end{array} \right. \tag{5.3}$$

Using equation (5.1), we firstly prove that the solutions of this system are necessarily of norm one. Taking into account all different cases, explicit computations lead to a contradiction with the system (5.3).

### Lemma

*$Y$ is a manifold with vanishing first Chern class.*

### Proof

In a first time, we shall give a "self-contained" proof, using the description of $Y$, given by equation (5.1). Then, in a second time, taking into account equation, we give a shorter proof which uses some concepts of Algebraic Geometry. The $c_1$ nobody depends on the choice of the degree of the hypersurface we introduce then the residue of Poincaré. It is about the generalization of the classic notion of residue of Cauchy in a point of a domain of $\mathbb{C}$ to that of residue in a hypersurface of a variety of complex dimension $n$. If $\eta$ is a méromorphe $n$-shape of such a variety which has excellent oles on a hypersurface $X$ defined locally by the equation $g = 0$, then $\eta$ can spell locally under the shape :

$$\eta = \frac{\gamma \wedge dg}{g} + \delta$$

where $\gamma$ and $\delta$ are respectively one $(n - 1)$ and one $n$ forms holomorphes. The limitation of $\gamma$ in $Y$ is then one $(n - 1)$ train(form) holomorphe defined well on $Y$

and called residue of Poincaré of $\eta$. In our case, let us consider the local writing in the card(map) $(U_{01}, \varphi_{01})$ defined higher :

$$\eta = \frac{d\xi_1 \wedge d\xi_2 \wedge d\xi_3 \wedge d\xi_4 \wedge d\xi_5 \wedge d\xi_6}{1 + \sum_{i=1} \xi_i^n + (\xi_1\xi_4 - \xi_2\xi_3)^n + (\xi_1\xi_6 - \xi_2\xi_5)^n + (\xi_3\xi_6 - \xi_4\xi_5)^n}.$$

Let us prove that the determinant bundle $\Lambda^6 T^*Y$ is trivializable by giving a $(5,0)$-holomorphic volume form $\Omega$ from a $(6,0)$-meromorphic form on $G_{2,5}\mathbb{C}$, using Poincaré residue. It corresponds to the generalization of the classical Cauchy residue at a point of a domain of $\mathbb{C}$ to the concept of residue in a hypersurface of a $n$ dimensional complex manifold. If $\eta$ is a $n$-meromorphic form of such a manifold which has first order poles on the hypersurface $Y$ locally defined by the equation $g = 0$, then $\eta$ may be locally written as:

$$\eta = \frac{\gamma \wedge dg}{g} + \delta,$$

where $\gamma$ and $\delta$ are respectively $(n-1)$ and $n$ holomorphic forms. Then, the restriction of $\gamma$ to $Y$ is well defined as a $(n-1)$ holomorphic form on $X$. We say that $\gamma$ is the Poincaré residue of $\eta$.

In our case, let us consider the open chart set $U_{01}$ of $G_{2,5}\mathbb{C}$ where the $\{U_{ij}\}_{0 \leq i < j \leq 3}$ are given by :

$$U_{ij} = \{(u,v) \in \mathbb{C}^4 \times \mathbb{C}^4 : u = (z_0, z_1, z_2, z_3, z_4), u' = (z_0', z_1', z_2', z_3', z_4'), z_i z_j' - z_j z_i' \neq 0\}.$$

The real place (the fixed points of an involution antiholomorphe) of a manifold (respectively orbifold) in first Chern class is a special Lagrangian manifold (respectively orbifold). indeed R.L. Bryant is shown in [2] that the real place of a manifold (respectively orbifold) of Calabi-Yau ($c_1 = 0$) is a special Lagrangian manifold. For $n = 6$:

### Theorem

*The real locus $L$ of $Y$ is a special Lagrangian submanifold endowed with a circle bundle structure.*

## 6. Genaralization of the result in $G_{p,p+q}\mathbb{C}$

**Definition of the hypersurface $Z$:**

$$K = \begin{pmatrix} z_{11} & . & . & . & z_{1p} \\ . & . & . & . & . \\ . & . & . & . & . \\ . & . & . & . & . \\ z_{(p+q)1} & . & . & . & z_{(p+q)p} \end{pmatrix}$$

The matrix of the p-vectors of $\mathbb{C}^{p+q}$, they define an espace and consequently an element of $G_{p,p+q}\mathbb{C}$ and $\Delta_i$ The minors of order $p$ of this matrix. We suggest studying the hypersurface defined by the following equation:

$$M(\Delta) = \sum_{i=1}^{C_{p+q}^{p}} \det{(\Delta_i)}^d = 0$$

where $d$ The degree of the hypersurface is to be determined so that its Chern class is useless.

**Lemma.** *The equation of the hypersurface $Z$ is intrinsic for any change of map in $G_{p,p+q}\mathbb{C}$ which can etre characterized by the intersection of cross-functional curves in $\mathbb{P}_{C_{p+q}^{p}-1}\mathbb{C}$.*

**Proof.** $A$ a matrix of change of map defined as follows:

$$\begin{pmatrix} a_{11} & . & . & . & a_{1p} \\ . & . & . & . & . \\ . & . & . & . & . \\ . & . & . & . & . \\ a_{p1} & . & . & . & a_{pp} \end{pmatrix} \in Gl_p(\mathbb{C})$$

And $\delta$ is the determining of $A$ by calculating $A.K$ and in replacement in the equation of $Z$ the new constituents let us obtain $M(\Delta') = \delta^d * \sum_{i=1}^{C_{p+q}^{p}} \det{(\Delta_i)}^d = 0$.
What proves that the writing of $Z$ is intrinsic.

Now looking at the writing of this hypersurface in the atlas defined by the following way: $U_i = \{(u_1, ..., u_p); u_1 = (z_{11}, ..., z_{1(p+q)}); u_p = (z_{p1}, ..., z_{p(p+q)}); det(\Delta_i) \neq 0\}$

$$\phi_1 = \begin{pmatrix} z_{11} & . & . & . & z_{1p} \\ . & . & . & . & . \\ . & . & . & . & . \\ . & . & . & . & . \\ z_{(p+q)1} & . & . & . & z_{(p+q)p} \end{pmatrix} . \begin{pmatrix} z_{11} & . & . & . & z_{1p} \\ . & . & . & . & \\ . & . & . & . & . \\ . & . & . & . & . \\ z_{1p} & . & . & . & z_{pp} \end{pmatrix}^{-1} = \begin{pmatrix} & & I_p & & \\ & & & & \\ y_1 & . & . & . & y_p \\ . & . & . & . & . \\ . & . & . & . & . \\ y_{(p-1)q} & . & . & . & y_{pq} \end{pmatrix}$$

where $I_p$ Indicate the matrix identity of order p;

$$M_{U_1,\phi_1} = 1 + \sum_{i=1}^{pq} y_i^d + \sum_{j=1}^{C_{p+q}^{p}-pq-1} \delta_i^d = 0$$

Où $\delta_i$ Indicate the determiners of the minors of order p of it under matrix $y_i$. $G_{p,p+q}\mathbb{C}$ plunges in $\mathbb{P}_{C_{p+q}^{p}-1}\mathbb{C}$ defining an intersection of cross-functional curves. With the equation $Z$ is completely defined as a hypersurface resulting from the intersection of the cross-functional curves even if it means determining its degree to obtain $c_1(Z) = 0$. For that we are going to use the residue of Poincaré:

$$\eta = \frac{dy_1 \wedge ... \wedge dy_{pq}}{1 + \sum_{i=1}^{pq} y_i^d + \sum_{j=1}^{C_{p+q}^p - pq - 1} \delta_i^d}$$

this writing what is prolongeable on all maps of $G_{p,p+q}\mathbb{C}$ that gives the $c_1(Z)$ is useless for an equal power $pq$ what confirms the result on $G_{2,5}\mathbb{C}$ where from $d = pq$ in some dimension thus of Calabi-Yau.

**Theorem.** *The real locus of $Z$ is:*

- *A special Lagrangian orbifold if $Z$ presents some singularities that depends of the dimension.*
- *A special Lagrangian manifold if $Z$ if there is no singularity.*

## References


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