International Journal of Research in Applied Sciences, 05 (2024) 232-239
Print ISBN: 2356-5675
Online ISSN: 2356-5705

www.knowledgejournals.com

Lower bound of plurisubharmonic functions in $G_{2,5}\mathbb{C}$ invariant by a group of automorphisms

Riadh JELLOUL

Faculty of Sciences of Tunis
Unit of geometry and non-linear analysis

E-mail: riadh.jelloul@gmail.com

Article history: Received 23 November 2023; Received in revised form 28 December 2023. Accepted $3rd$ 2024; Available online $5th$ January 2024.

Abstract. In a previous work [1], I proved the existence of a function minimizing all functions admissible to sup zero on a non-toric algebraic variety namely $G_{2,4}\mathbb{C}$. The functions considered are invariant by a group of automorphisms obtained from that of $\mathbb{P}_m(\mathbb{C})$. We prove in this paper that the Tian invariant on the Grassmannian $G_{2,5}\mathbb{C}$ is 5/18. The method presented in this article uses a group of automorphisms which operates transitively on $G_{2,5}\mathbb{C}$ as well as a dip natural of it in $\mathbb{P}_9\mathbb{C}$.

Keywords: Grassmann manifold, plurisubharmonic function, Tian's invariant.

2000 Mathematics subject classification: 51F05, 51F10, 51K05.

Introduction

On a complex manifold, an hermitian metric h is characterized by the *1-1 symplectic* form ω defined by $\omega = i\, g_{\lambda\bar{\mu}}\, dz^{\lambda} \wedge d\bar{z}^{\mu}$, where $g_{\lambda\bar{\mu}} = h_{\lambda\bar{\mu}}/2$.

The metric is a *Kähler metric* if ω is closed, i. e. $d\omega = 0$; then M is a *Kähler manifold.*

On a Kähler manifold, we can define the *Ricci form* by $R = i\, R_{\lambda\bar{\mu}}\, dz^{\lambda} \wedge d\bar{z}^{\mu}$, where $R_{\lambda\bar{\mu}} = -\partial_{\lambda\bar{\mu}} \log |g|$.

A Kähler manifold is *Einstein with factor* k if $R = k\omega$. For instance, choosing a local coordinate system $Z = (z_1, \dots, z_m)$, the projective space $\mathbb{P}_m\mathbb{C}$ with the Fubini-Study metric $\omega = i\partial\bar{\partial}\log\big(1 + ||Z||^2\big)$ is Einstein with factor $m + 1$.

On a Kähler manifold M, the *first Chern class* $C^1(M)$ is the cohomology class of the Ricci tensor, that is the set of the forms $R + i\partial\bar{\partial}\varphi$, where φ is C^∞ on M. If there is a form in $C^1(M)$ which is positive (resp. negative, zero), then $C^1(M)$ is *positive* (resp. *negative*, *zero*). If a Kähler manifold is Einstein, then $C^1(M)$ and k are both positive (resp. negative, zero). In the negative case, it was proved by Aubin ([Au1], see also [Au4]), that there exists a unique Einstein-Kähler metric (E.K. metric) on M . It is so for the zero case too ([Au1], [Ya]). The question for the positive case is still open: some manifolds, such as the complex projective space blown up at one point, do not admit an E.K. metric (for obstructions, see [Li] and [Fu]). Aubin [Au2] and Tian [Ti] have shown that for suitable values of holomorphic invariants of the metric, there exists an E.K. metric on M.

For $\omega/2\pi$ in $C^1(M)$, *Tian's invariant* $\alpha(M)$ is the supremum of the set of the real numbers α satisfying the following: there exists a constant C such that the inequality $\displaystyle\int_M e^{-\alpha\varphi} \leq C$ holds for all the C^∞ functions φ with $\omega + i\partial\bar{\partial}\varphi > 0$ and $\sup\varphi \geq 0$, where $\omega = i\, g_{\lambda\bar{\mu}}\, dz^\lambda \wedge d\bar{z}^{\,\mu}$ is the metric form. Such functions φ are said ω*-admissible*.

In [Ti], Tian established that if $\alpha(M) > m/(m + 1)$, m being the dimension of M, there exists an E.K. metric on M. This condition is not necessary: it does not hold on the projective space, where Tian's invariant is $1/(m + 1)$.

In the same paper, Tian introduces a more restrictive invariant $\alpha_G(M)$, considering only the admissible functions φ invariant by the action of a compact group G of holomorphic isometries. The sufficient condition for the existence of an E.K. metric on M remains $\alpha_G(M) > m/(m + 1)$; it is more easily satisfied if the group G is rich enough.

1. Results obtained on $G_{2,5}\mathbb{C}$:

Let $(G_{2,5}\mathbb{C}, g)$, the complex Grassmannian of the two planes of $\mathbb{C}^5$ provides the metric g, obtained from that of Fubini-Study on $\mathbb{P}_9\mathbb{C}$, and belonging to the first Chern class, $c_1(G_{2,5}\mathbb{C})$.
We locate the points of $G_{2,5}\mathbb{C}$, by a matrix of $M_{5,2}(\mathbb{C})$:

$$\begin{pmatrix} z_0 & z'_0 \\ z_1 & z'_1 \\ z_2 & z'_2 \\ z_3 & z'_3 \\ z_4 & z'_4 \end{pmatrix}$$

where the two column vectors are independent. The card openers $U_{i,j}$, $0 \leq i < j \leq 4$ are obtained by considering the minor of order 2 relating to the lines i, j with a non-zero determinant. For example, in $U_{0,1}$, a point of the Grassmannian is written:

$$\begin{pmatrix} 1 & 0 \\ 0 & 1 \\ z_1 & z_2 \\ z_3 & z_4 \\ z_5 & z_6 \end{pmatrix},$$

and the metric g has components in this map: $g_{\lambda\bar{\mu}} = 5\partial_{\lambda\bar{\mu}} \ln(1+ \mid z_1 \mid^2 + \mid z_2 \mid^2 + \mid z_3 \mid^2 + \mid z_4 \mid^2 + \mid z_5 \mid^2 + \mid z_6 \mid^2 + \mid z_1 z_4 - z_2 z_3 \mid^2 + \mid z_1 z_6 - z_2 z_5 \mid^2 + \mid z_3 z_6 - z_4 z_5 \mid^2)$ (the constant 5 is the one which ensures that the metric is indeed in the c_1, see [2]), where $\partial_{\lambda\bar{\mu}} = \frac{\partial^2}{\partial z_\lambda \partial \bar{z}_\mu}$.

We say that a function $\varphi \in C^\infty(G_{2,5}\mathbb{C})$ is g-admissible, if $g_{\lambda\bar{\mu}} + \partial_{\lambda\bar{\mu}}\varphi$ is positive definite (it therefore defines a new metric).

Now consider the function $\tilde{\psi}$, defined in $U_{i,j}$ by : $\tilde{\psi}\begin{pmatrix} z_0 & z'_0 \\ z_1 & z'_1 \\ z_2 & z'_2 \\ z_3 & z'_3 \\ z_4 & z'_4 \end{pmatrix} =$

$\ln \frac{|z_0 z'_1 - z'_0 z_1|^{10/6} |z_0 z'_3 - z'_0 z_3|^{10/6} |z_1 z'_2 - z'_1 z_2|^{10/6} |z_1 z'_4 - z'_1 z_4|^{10/6} |z_2 z'_3 - z'_2 z_3|^{10/6} |z_3 z'_4 - z'_3 z_4|^{10/6}}{(\sum_{0 \le i < j \le 3} |z_i z'_j - z'_i z_j|^2)^5}$. This function, of ten complex variables, is independent of the choice of the representative of the plane of $\mathbb{C}^5$, it therefore defines a function on $G_{2,5}\mathbb{C}$ (deprived of the edge of the cards).

We set $\psi = \tilde{\psi} - \sup \tilde{\psi} = \tilde{\psi} + 5\ln(6)$. ψ is negative and admits zero sup at the point $\begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 1 & 0 \\ 0 & 1 \\ 1 & 0 \end{pmatrix}$ of the map $U_{0,1}$ (see proposition 4).

Finally, consider the group G, generated by the automorphisms $\phi_{\begin{pmatrix} a & b \\ c & d \end{pmatrix}}$ and P of $G_{2,5}\mathbb{C}$, defined, for $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in U_2(\mathbb{C})$, by :

$$\phi_{\begin{pmatrix} a & b \\ c & d \end{pmatrix}} \begin{pmatrix} z_0 & z'_0 \\ z_1 & z'_1 \\ z_2 & z'_2 \\ z_3 & z'_3 \\ z_4 & z'_4 \end{pmatrix} = \begin{pmatrix} az_0 + bz_1 & az'_0 + bz'_1 \\ cz_0 + dz_1 & cz'_0 + dz'_1 \\ z_2 & z'_2 \\ z_3 & z'_3 \\ z_4 & z'_4 \end{pmatrix}$$

The invariance of a function f by P_i which generates the permutation of the first two blocks of order 2 and the permutation of line 1 with line 5. We easily verify that these applications are intrinsic and leave the metric g invariant. G is therefore a group of isometries of $G_{2,5}\mathbb{C}$, in the sense of the metric g. We also check that the function ψ, defined above is G-invariant.

In this article we show that the functions $\varphi \in C^\infty(G_{2,5}\mathbb{C})$, g-admissible, with sup equal to zero on $G_{2,5}\mathbb{C}$, invariant by G, are reduced by the function $\tilde{\psi}$ (which we will henceforth call "extremal function"), a function tending towards minus infinity on the edge of the maps usual (described above) of $G_{2,5}\mathbb{C}$.

A similar reduction has been proven on the complex projective as well as on varieties obtained from the latter by splitting and by fibration, a reduction which provides a Tian type inequality on these toric manifolds, and makes it possible to establish lower bounds of their Ricci tensors.
This article deals with a non-toric example, using a method other than that recommended in [2], adding a group of automorphisms making it possible to find a larger Tian constant. Let us now state the main results of this article.

Théorème 1. *Let $\varphi \in C^\infty(G_{2,5}\mathbb{C})$ be a function g-admissible and G-invariant, checking* $\sup_{G_{2,5}\mathbb{C}} \varphi = \varphi \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 1 & 0 \\ 0 & 1 \\ 1 & 0 \end{pmatrix} = 0$. *Then we have: $\varphi \geq \psi$.*

Théorème 2. *Let $\varphi \in C^\infty(G_{2,5}\mathbb{C})$ be a function g-admissible and G-invariant, checking* $\sup_{G_{2,5}\mathbb{C}} \varphi = 0$. *Then we have: $\varphi \geq \tilde{\psi}$.*

A corollary of this theorem is:

Théorème 3. . *For all $\alpha < 5/18$, we have the inequality following Tian type (see* [4]*:*

$$\int_{G_{2,5}\mathbb{C}} \exp(-\alpha\varphi)dv \leq Cst,$$

for any function φ verifying the hypotheses of the theorem 1.
dv is the volume element on $G_{2,5}\mathbb{C}$ relative à to the metric g.

2. Proofs of Theorems 1 and 2

All the calculations which will follow are carried out in the intersection of the cards $U_{i,j}$.

2.1. Proof of Theorem 1.

Group of automorphisms in $G_{2,5}\mathbb{C}$. We define the group of isometries in $G_{2,5}\mathbb{C}$: Let X be the set of subspaces of dimension 2 in $\mathbb{C}^5$; especially, $G_{1,m}\mathbb{C}$ is the complex projective space of dimension m. It is known that on $\mathbb{P}_m\mathbb{C}$, the Fubini-Study metric is Einstein with the factor $m+1$ and that the Tian invariant is $1/(m+1)$. Now, let $M_2\mathbb{C}$ be set of matrices of rank 2. The $Gl_2\mathbb{C}$ group acts by right multiplication on $M_2\mathbb{C}$. More precisely the group $Gl_5\mathbb{C}$ acts by multiplication left on $M_2\mathbb{C}$ and induces an action on $G_{2,5}\mathbb{C}$; the same goes for the unitary group $U_5\mathbb{C}$. These groups act transitively on $G_{2,5}\mathbb{C}$, which shows that $G_{2,5}\mathbb{C}$ is compact. We denote by I the set of all subsets of increasing order of 2 items in $\{1,\ldots,5\}$. Let R be an element of $M^*_{5,2}\mathbb{C}$, $R = \left(r_{ij}\right)_{\substack{1\leq i\leq 5\\ 1\leq j\leq 2}}$. By the Cauchy-Binet formula we obtain: $\det\left({}^t R\overline{R}\right) = \sum_I |\det m_{_I}(R)|^2$, where $m_I(R)$ is the matrix $\left(r_{ij}\right)_{\substack{i\in I\\ 1\leq j\leq 2}}$. The form ω, Or $\omega = i\,\partial\overline{\partial}\,\log\det\left({}^{t}R\overline{R}\right)$, is invariant by the action of $Gl_2\mathbb{C}$ on V, and therefore it projects onto a form $G_{2,5}\mathbb{C}$. The metric g is a Kähler metric on $G_{2,5}\mathbb{C}$. The action of the unitary group $U_5\mathbb{C}$ on $G_{2,5}\mathbb{C}$ preserves the metric g so that $U_5\mathbb{C}$ is a group of holomorphic isometries which operate transitively on $G_{2,5}\mathbb{C}$.

Proposition 4. *ψ is G-invariant and reaches its sup, equal to zero, on $U_5\mathbb{C}$ (which amounts to saying that $\tilde{\psi}$ reaches its sup, equal to $-5\ln(6)$, on U_5). ψ and $\tilde{\psi}$ verify:*

$$\partial_{\lambda\bar{\mu}}\psi = \partial_{\lambda\bar{\mu}}\tilde{\psi} = -g_{\lambda\bar{\mu}}$$

Lemme 1. *Let $\lambda_1 \neq 1$ and $\lambda_2 \neq 1$ be two strictly positive real numbers and let φ be a function verifying the hypotheses of theorem 1. Then:*

$$(\varphi-\psi)\begin{pmatrix} 1 & 0 \\ 0 & \lambda_1 \\ \lambda_2 & 0 \end{pmatrix} \geq 0$$

and

$$(\varphi-\psi)\begin{pmatrix} 0 & \lambda_1 \\ \lambda_2 & 0 \\ 0 & 1 \end{pmatrix} \geq 0.$$

(writing in card $U_{0,1}$).

Proof.

Let us proceed absurdly by assuming that there exists $\lambda_1 > 0$ and $\lambda_2 > 0$, $\lambda_1 \neq 1$ and $\lambda_2 \neq 1$ such that:

$$(\varphi-\psi)\begin{pmatrix} 1 & 0 \\ 0 & \lambda_1 \\ \lambda_2 & 0 \end{pmatrix} < 0$$

same proof for $\begin{pmatrix} 0 & \lambda_1 \\ \lambda_2 & 0 \\ 0 & 1 \end{pmatrix}$).

Let the set be defined for $\lambda_1 > 1$ and $\lambda_2 > 1$by :

$$D_\lambda = \bigcup_{t\in[-1,1]} \{ \bigcup_{\theta\in[0,2\pi]} \begin{pmatrix} 1 & 0 \\ 0 & \lambda_1^t e^{i\theta_1} \\ \lambda_2^t e^{i\theta_2} & 0 \end{pmatrix}$$

Therefore, using the fact that $\varphi - \psi$ is G-invariant, we have:

$$(\varphi-\psi)\begin{pmatrix} 1 & 0 \\ 0 & \lambda_1^t e^{i\theta_1} \\ \lambda_2^t e^{i\theta_2} & 0 \end{pmatrix} = (\varphi-\psi)\begin{pmatrix} 1 & 0 \\ 0 & \lambda_1^t \\ \lambda_2^t & 0 \end{pmatrix}.$$

Knowing that $\varphi - \psi$ is G-invariant, it is invariant in particular by the inversion of the matrices (expression in $U_{0,1} \cup U_{2,3}$ of the invariance of the P automorphisms described above). The function $\varphi - \psi$ therefore takes the same value on C_t as on C_{-t}. It is therefore negative on the edge $C_{-1} \cup C_1$ of the aforementioned crown and identically zero on the circle C_0 (the functions φ and ψ are zero on the circle C_0 corresponding to the orbit of the identity matrix). It therefore reaches its sup inside the crown which we will parameterize, in the map $U_{0,1}$, by the holomorphic curve :

$$c(z) = (c^1(z) = 1, c^2(z) = 0, c^3(z) = 0, c^4(z) = z_1, c^5(z) = z_2, c^6(z) = z_2)$$

At a point z_0 interior to the crown where the sup is reached, we have :

$$\frac{\partial^2[(\varphi-\psi)(c(z))]}{\partial z\partial\bar{z}}(z_0) = \frac{\partial^2(\varphi-\psi)}{\partial z^i\partial\bar{z}^j}(c(z_0))\dot{c}^i(z_0)\dot{\bar{c}}^j(z_0) < 0,$$

which contradicts the admissibility hypothesis of φ.

Lemme 2. *Let λ_1, λ_2 and λ_3 be 3 strictly positive real numbers and different from 1, and let φ be a function verifying the hypotheses of the theorem. Then we have:*

$$(\varphi-\psi)\begin{pmatrix}\lambda_1 & 0\\ 0 & \lambda_2\\ \lambda_3 & 0\end{pmatrix} \geq 0$$

(writing in card $U_{0,1}$).

Proof. As in the previous lemma, we will reason through the absurd by assuming that there exists $0<\lambda_1$, $0<\lambda_2$ and $0<\lambda_3$ both different from 1 , such that

$$(\varphi-\psi)\begin{pmatrix}\lambda_1 & 0\\ 0 & \lambda_2\\ \lambda_3 & 0\end{pmatrix} < 0.$$

Note that the invariance of $(\varphi-\psi)$ by $U_2(\mathbb{C})$ implies:

$$(\varphi-\psi)\begin{pmatrix}\lambda_1^t & 0\\ 0 & \lambda_2^t\\ \lambda_3^t & 0\end{pmatrix} = (\varphi-\psi)\begin{pmatrix}\lambda_1^t e^{i\theta_1} & 0\\ 0 & \lambda_2^t e^{i\theta_2}\\ \lambda_3^t e^{i\theta_3} & 0\end{pmatrix}.$$

And

C is a complex domain of complex dimension 3 of the map $U_{0,1}$ whose edge is given by the orbits by the action of $A\subset U_2(\mathbb{C})$:

$$\begin{pmatrix}\zeta_1 & 0\\ 0 & \zeta_2\\ \zeta_3 & 0\end{pmatrix} \text{ with } \frac{1}{\lambda_1} < |\zeta_1| < \lambda_1, \frac{1}{\lambda_2} < |\zeta_2| < \lambda_2 \text{ and } \frac{1}{\lambda_3} < |\zeta_3| < \lambda_3.$$

We can therefore identify the points of C by (ζ_1, ζ_2), where $\frac{1}{\lambda_1} < |\zeta_1| < \lambda_1$, $\frac{1}{\lambda_2} < |\zeta_2| < \lambda_2$ and $\frac{1}{\lambda_3} < |\zeta_3| < \lambda_3$. C therefore describes a product of tori whose edge is given by:

$$\partial C = \{(\zeta_1,\zeta_2,\zeta_3)\in\mathbb{C}^3; \zeta_1 = \lambda_1 \text{ or } \frac{1}{\lambda_1}, \text{and} \zeta_2 = \lambda_2 \text{ or } \frac{1}{\lambda_2} and \zeta_3 = \lambda_3 \text{ or } \frac{1}{\lambda_3},$$

where the function $(\varphi-\psi)$ is strictly negative, according to the initial hypothesis and the fact that $(\varphi-\psi)$ is G-invariant. Indeed, like the previous lemma, the invariance of a function by G results in the fact that this function remains constant on $E_t \cup E_{-t}$. Consider the curve:

$$c(t) = \begin{pmatrix} t & 0 \\ 0 & t^{\frac{\ln \lambda_2}{\ln \lambda_1}} \\ t^{\frac{\ln \lambda_3}{\ln \lambda_1}} & 0 \end{pmatrix},$$

defined on $[\lambda_1, \frac{1}{\lambda_1}]$ (we can assume, even if it means reversing the roles of λ_1 and $\frac{1}{\lambda_1}$), that $\lambda_1 < 1$. The curve $c(t)$ passes, respectively in $t = \lambda_1$, $t = 1$ and $t = \frac{1}{\lambda_1}$ through the points $\begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \\ \lambda_3 & 0 \end{pmatrix}$, $\begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 1 & 0 \end{pmatrix}$ and $\begin{pmatrix} \lambda_1^{-1} & 0 \\ 0 & \lambda_2^{-1} \\ \lambda_3^{-1} & 0 \end{pmatrix}$, where $\varphi - \psi$ is respectively strictly negative, zero, then strictly negative again. Knowing that $\varphi - \psi$ is G-invariant, we have:

$$\begin{aligned} (\varphi-\psi)(c(t)) &= (\varphi-\psi)\begin{pmatrix} t & 0 \\ 0 & t^{\frac{\ln \lambda_2}{\ln \lambda_1}} \\ t^{\frac{\ln \lambda_3}{\ln \lambda_1}} & 0 \end{pmatrix} \\ &= (\varphi-\psi)\begin{pmatrix} te^{i\theta} & 0 \\ 0 & (te^{i\theta})^{\frac{\ln \lambda_2}{\ln \lambda_1}} \\ (te^{i\theta})^{\frac{\ln \lambda_3}{\ln \lambda_1}} & 0 \end{pmatrix} \\ &= (\varphi-\psi)\begin{pmatrix} z & 0 \\ 0 & z^{\frac{\ln \lambda_2}{\ln \lambda_1}} \\ z^{\frac{\ln \lambda_3}{\ln \lambda_1}} & 0 \end{pmatrix} \\ &= (\varphi-\psi)(c(z)), \end{aligned}$$

defined in the crown : $\{\lambda_1 \leq |z| \leq \frac{1}{\lambda_1}\}$ of $\mathbb{C}$. $(\varphi - \psi)(c(z))$ therefore admits a local maximun inside the crown described above (more precisely, at the points z such that $|z| = 1$). Its Hessian is therefore negative at these points, which allows us to write, as in the previous lemma: Consequently $(\varphi - \psi)$ admits a maximum inside the domain C of $\mathbb{C}^3$, which we will parameterize in $U_{i,j}$ by : $c(\zeta_1, \zeta_2, \zeta_3) = (c_1(\zeta_1, \zeta_2, \zeta_3), c_2(\zeta_1, \zeta_2, \zeta_3), c_3(\zeta_1, \zeta_2, \zeta_3), c_4(\zeta_1, \zeta_2, \zeta_3), c_5(\zeta_1, \zeta_2, \zeta_3)c_6(\zeta_1, \zeta_2, \zeta_3))$. The Hessian is therefore negative at a point interior to the domain C parameterized by

$$\frac{\partial^2(\varphi-\psi)}{\partial z^i \partial \bar{z}^j(c(z_0)})\dot{c}^i(z_0)\dot{\bar{c}}^j(z_0) < 0.$$

This contradicts the admissibility of φ.

3. Proof of Theorem 3

. According to the previous theorem we have:

$$\int\limits_{G_{2,5}\mathbb{C}} e^{-\alpha\varphi} dv \leq \int\limits_{G_{2,5}\mathbb{C}} e^{-\alpha\psi} dv$$

If $|g|$ denotes the determinant of the metric, $\ln(1+\mid z_1\mid^2+\mid z_2\mid^2+\mid z_3\mid^2+\mid z_4\mid^2+\mid z_5\mid^2+\mid z_6\mid^2+\mid z_1z_4-z_2z_3\mid^2+\mid z_1z_6-z_2z_5\mid^2+\mid z_3z_6-z_4z_5\mid^2)$ is then an intrinsic quantity. Indeed, this is due to the fact that $g_{\lambda\bar{\mu}}=5\partial_{\lambda\bar{\mu}}\ln(1+\mid z_1\mid^2+\mid z_2\mid^2+\mid z_3\mid^2+\mid z_4\mid^2+\mid z_5\mid^2+\mid z_6\mid^2+\mid z_1z_4-z_2z_3\mid^2+\mid z_1z_6-z_2z_5\mid^2+\mid z_3z_6-z_4z_5\mid^2)$. There are therefore two constants C_1 and C_2 such that: $C_1\leq\ln(1+\mid z_1\mid^2+\mid z_2\mid^2+\mid z_3\mid^2+\mid z_4\mid^2+\mid z_5\mid^2+\mid z_6\mid^2+\mid z_1z_4-z_2z_3\mid^2+\mid z_1z_6-z_2z_5\mid^2+\mid z_3z_6-z_4z_5\mid^2)|g|\leq C_2$.

The convergence of the last integral is therefore equivalent to the convergence of: $\int_{\mathbb{C}^6}e^{-\alpha\psi}\frac{dz_1\wedge d\bar{z}_1\wedge dz_2\wedge d\bar{z}_2\wedge dz_3\wedge d\bar{z}_3\wedge dz_4\wedge d\bar{z}_4\wedge dz_5\wedge d\bar{z}_5\wedge dz_6\wedge d\bar{z}_6}{\ln(1+|z_1|^2+|z_2|^2+|z_3|^2+|z_4|^2+|z_5|^2+|z_6|^2+|z_1z_4-z_2z_3|^2+|z_1z_6-z_2z_5|^2+|z_3z_6-z_4z_5|^2}$, and taking into account the invariances by G of the functions considered, this amounts to studying the convergence of : $\int_0^{+\infty}\int_0^{+\infty}\int_0^{+\infty}\frac{(uvw)^{-10/6\alpha}dudvdw}{(1+u+v+uv+vw)^{6(1-\alpha)}}$ which takes place as soon as $\alpha<5/18$.

Tian invariant and perspectives

The Tian invariant obtained is equal to 5/18, at this stage we have improved the result of J. Grivaux see [2]. The work is in progress and this depends on the enrichment of the group of automorphisms in order to expect an invariant equal to 1 which results in the existence of Einstein-Käler metrics in the Grassmannian and generalization in any dimensions.

References

Au1. T. Aubin, *Équations du type Monge-Ampère sur les variétés kählériennes compactes*, Bull. Sci. Math. **102** (1978), 63-95.

Au2. T. Aubin, *Réduction du cas positif de l'équation de Monge-Ampère sur les variétés Kählériennes compactes à la démonstration d'une inégalité*, J. Funct. Anal. **57** (1984), 143-153.

Au3. T. Aubin, *Métriques d'Einstein-Kähler et exponentiel de fonctions admissibles*, J. Func. Anal. **88** (1990), 385-394.

Au4. T. Aubin, *Some Non-linear Problems in Riemannian Geometry*, Springer-Verlag, Berlin, 1998.

Fu. A. Futaki, *An obstruction to the existence of Kähler-Einstein metrics*, Invent. Math. **73**, 1983, 437-443.

Ko-No. S. Kobayashi, K. Nomizu, *Foundations of Differential Geometry*, Volume II John Wiley (1969).

Li. A. Lichnerowicz, *Sur les transformations analytiques des variétés kählériennes*, Cr. Acac. Sci **244**, 1957, 3011-3014.

Ma. Y. Matsushima, *Sur la structure du groupe d'homéomorphismes analytiques d'une certaine variété kählérienne*, Nagoya Math. J. **11** (1957), 145-150.

Re. C. Real, *Métriques d'Einstein-Kähler sur des variétés à première classe de Chern positive*, J. Func. Anal. **106** (1992), 145-188.

Ti. G. Tian, *On Kähler-Einstein metrics on certain Kähler manifolds with $C^1(M)>0$*, Invent. Math. **89** (1987), 225-246.

Ya. S. T. Yau, *On the Ricci curvature of a compact Kähler manifold and the complex Monge-Ampère equations*, I. Comm. Pure Appl. Math. **31** (1978), 339-411.

1. A. Ben Abdesselem and R. Jelloul– Enveloppe inférieure de fonctions admissibles sur la Grassmannienne $G_{2,4}\mathbb{C}$ en présence de symétries, *Bulletin des Sciences Mathématiques (2013), 139-146.*

2. J. Grivaux–Tian constant on Grassmann manifolds. Kähler-Einstein metrics, *J. of geometric Analysis. Vol.16 , 2006, 523-533.*
3. L. Hőrmander – An introduction to complex analysis in several variables, *North-Holland, Amsterdam, 1973.*
4. G. Tian– On Kähler-Einstein metrics on certain Kähler manifolds with $c_1(M) > 0$, *Invent. Math. 89, 1987, 225-246.*